On the almost Gorenstein property of determinantal rings

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Introduction

Setting 1.1

- $2 \leq t \leq m \leq n$ integers
- $X = [X_{ij}]$ an $m \times n$ matrix of indeterminates over an infinite field k
- $S = k[X] = k[X_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ the polynomial ring
- $\mathrm{I}_t(X)$ the ideal of S generated by the $t \times t$ minors of the matrix X
- $R = S/\mathrm{I}_t(X)$

Then R is a Cohen-Macaulay normal domain and

$$R \text{ is Gorenstein} \iff m = n.$$

History of almost Gorenstein rings

- [Barucci-Fröberg, 1997]

  $\cdots$ one-dimensional analytically unramified local rings

- [Goto-Matsuoka-Phuong, 2013]

  $\cdots$ one-dimensional Cohen-Macaulay local rings

- [Goto-Takahashi-T, 2015]

  $\cdots$ higher-dimensional Cohen-Macaulay local/graded rings

Question 1.2

When do the determinantal rings satisfy almost Gorenstein property?

Preliminaries

Setting 2.1

- $(A, \mathfrak{m})$ a Cohen-Macaulay local ring with $d = \dim A$
- $|A/\mathfrak{m}| = \infty$
- $\exists$ K_A the canonical module of A

Definition 2.2 (Goto-Takahashi-T, 2015)

We say that A is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to A \to \mathrm{K}_A \to C \to 0$$

of A-modules such that $\mu_A(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

Setting 2.3

- $R = \bigoplus_{n \geq 0} R_n$ a Cohen-Macaulay graded ring with $d = \dim R$
- $(R_0, \mathfrak{m})$ a local ring s.t. $|R_0/\mathfrak{m}| = \infty$
- $\exists$ K_R the graded canonical module of R
- $M = \mathfrak{m}R + R_+$

Definition 2.4 (Goto-Takahashi-T, 2015)

We say that R is *an almost Gorenstein graded ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R(-\mathrm{a}(R)) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_M(C)$.

Note that

- R is an almost Gorenstein graded ring
  $\Longrightarrow R_M$ is an almost Gorenstein local ring.

  $\Longleftarrow$ NOT true in general.

Example 2.5 (Goto-Takahashi-T, 2015)

Let k be an infinite field and

$$R = k[s, s^3t, s^3t^2, s^3t^3] \subseteq U$$

where $U = k[s, t]$ be the polynomial ring over k.

Note that

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  $\Longrightarrow$ R_M is an almost Gorenstein local ring.

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Theorem 2.6 (Goto-Takahashi-T, 2015)

Let $R = k[R_1]$ be a Cohen-Macaulay homogeneous ring with $d = \dim R > 0$. Suppose that R is not a Gorenstein ring and $|k| = \infty$. Then TFAE.

(1) *R is an almost Gorenstein graded ring and level.*

(2) *$\mathrm{Q}(R)$ is a Gorenstein ring and $\mathrm{a}(R) = 1 - d$.*

From now on, we maintain the notation as in Setting 1.1.

Corollary 2.7

$R = S/\mathrm{I}_t(X)$ is an almost Gorenstein graded ring $\iff$ $m = n$, or $m \neq n$ and $m = t = 2$.

Set $M = R_+$. Then

$$R = k[X]/\mathrm{I}_t(X) : AGG \implies R_M = (k[X]/\mathrm{I}_t(X))_M : AGL$$
$$\iff k[[X]]/\mathrm{I}_t(X) : AGL$$

Theorem 2.8

The following conditions are equivalent.

(1) *R is an almost Gorenstein graded ring.*

(2) *R_M is an almost Gorenstein local ring.*

(3) *Either $m = n$, or $m \neq n$ and $m = t = 2$.*

Proof of Theorem 2.8

For a moment, suppose that $\mathrm{ck}\, k = 0$ and $m \neq n$. Let

$$0 \to F \to G \to \cdots \to S \to R \to 0 \qquad (\sharp)$$

be a graded minimal S-free resolution of R.

Let

$$\alpha = \frac{\prod_{j=0}^{n-m-1} \left(\prod_{i=1}^{m-t} (t+i+j) \right) \prod_{i=0}^{n-m-2} (t+i) \cdot 1! \cdot 2! \cdots (m-t-1)! \cdot (m-t)!}{(n-m-1)! \cdot (n-m+1)! \cdot (n-m+2)! \cdots (n-t-1)! \cdot (n-t)!}.$$

Proposition 3.1

$$\mathrm{r}(R) = \mathrm{rank}_S F = \frac{t+n-m-1}{n-m} \cdot \alpha, \quad \mathrm{rank}_S G = n \cdot (m-t+1) \cdot \alpha$$

Take the K_S-dual of ($\sharp$), we get the presentation of K_R, which yields that

$$\mu_R(M\,\mathrm{K}_R) \geq mn \cdot \mathrm{r}(R) - \mathrm{rank}_S\, G.$$

Remark 3.2

Note that the Hilbert series of R doesn't depend on the field, so is the Hilbert series of K_R. Hence $\mu_R(M\,\mathrm{K}_R)$ doesn't depend on $\mathrm{ch}\, k$.

Theorem 2.8

The following conditions are equivalent.

(1) *R is an almost Gorenstein graded ring.*

(2) *R_M is an almost Gorenstein local ring.*

(3) *Either $m = n$, or $m \neq n$ and $m = t = 2$.*

Proof of Theorem 2.8

We may assume $m \neq n$. Since $A = R_M$ is an almost Gorenstein local ring, $\exists$ an exact sequence

$$0 \to A \to \mathrm{K}_A \to C \to 0$$

of A-modules s.t. $\mu_A(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$, where $\mathfrak{m} = MR_M$.

Then

$$0 \to \mathfrak{m} \to \mathfrak{m}\,\mathrm{K}_A \to \mathfrak{m}C \to 0$$

whence

$$\begin{aligned} \mu_A(\mathfrak{m}\,\mathrm{K}_A) &\leq \mu_A(\mathfrak{m}) + \mu_A(\mathfrak{m}C) \\ &\leq mn + (d-1)(\mathrm{r}(A) - 1) \end{aligned}$$

because $\mathfrak{m}C = (f_1, f_2, \ldots, f_{d-1})C$ for $\exists\ f_i \in \mathfrak{m}$, where $d = \dim A$.

Proof of Theorem 2.8

Therefore

$$mn \cdot \mathrm{r}(A) - \mathrm{rank}_S G \le \mu_A(\mathfrak{m}\,\mathrm{K}_A) \le mn + (d-1)(\mathrm{r}(A)-1)$$

which yields that

$$(mn - (d-1))\,(\mathrm{r}(A)-1) \le \mathrm{rank}_S G.$$

Hence

$$\{(m-(t-1))(n-(t-1))+1\}\left(\frac{t+n-m-1}{n-m}\cdot\alpha - 1\right) \le n(m-(t-1))\alpha.$$

Then a direct computation shows that $t = 2$, whence $m = 2$ as desired. $\square$

Thank you so much for your attention.